Theoretical Astrophysics

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1 Derivation of the virial theorem

Suppose we have a large system of particles in hydrostatic equilibrium. We will define what we mean by "hydrostatic" soon. We want to derive the relationship between the time-averaged kinetic and potential energies. We start by considering the following quantity:

$$Q = \sum_i \mathbf{r}_i \cdot \mathbf{p}_i, \tag{1}$$

where $\mathbf{r}_i$ is the i-th particle position and $\mathbf{p}_i$ is its momentum. We should not concern ourselves with the physical meaning of Q just yet, we will get there next. Let us evaluate a time derivative:

$$\dot{Q} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \mathbf{r}_i \cdot \dot{\mathbf{p}}_i. \tag{2}$$

Noting that $\mathbf{p}_i = m_i v_i$, the expression becomes:

$$\dot{Q} = \sum_i m_i \dot{r}_i^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \sum_i 2K_i + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i. \tag{3}$$

Now comes the time-averaging part. To evaluate a time average on a function $f(t)$, we need to calculate the following integral:

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t) dt \tag{4}$$

for some large enough T that would cause particles to traverse the available space many times. In effect,

$$\langle f(t) \rangle = \lim_{T \to \infty} \int_0^T f(t) dt. \tag{5}$$

We now apply this to our expression:

$$\langle \dot{Q} \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \dot{Q}dt = \lim_{T\to\infty}\frac{1}{T}[Q(T) - Q(0)] = \sum_i 2\langle K_i\rangle + \sum_i \langle \mathbf{F}_i \cdot \mathbf{r}_i\rangle. \quad (6)$$

The question now is whether $Q(t)-Q(0)$ is finite; if so, then the left-hand side will equal 0. That is what we mean when we say "in hydrostatic equilibrium." Then we get:

$$2\sum_i \langle K_i\rangle \equiv 2\langle K\rangle = -\sum_i \langle \mathbf{F}_i \cdot \mathbf{r}_i\rangle. \quad (7)$$

This is already the virial theorem even though it might not be obvious yet. We can put this equation to a test; imagine we have a box of ideal monoatomic gas and we want to evaluate the virial theorem. According to the equipartition theorem, $K_i = \frac{3}{2}kT$, so the total (time-averaged) kinetic energy is:

$$\langle K\rangle = \frac{3}{2}NkT. \quad (8)$$

Applying the virial theorem:

$$2\langle K\rangle = 3NkT = -\sum_i \langle \mathbf{F}_i \cdot \mathbf{r}_i\rangle. \quad (9)$$

As the gas is ideal, there are no interactions between the particles, but there is interaction with the walls. A single bounce contribution is:

$$d\mathbf{F}_i = -pd\mathbf{A} \quad \Rightarrow \quad \mathbf{F}_i = -\int_{\partial V} p\mathbf{r}_i \cdot d\mathbf{A} = -\int_V p\nabla \cdot \mathbf{r}_i \, dV. \quad (10)$$

As $\nabla \cdot \mathbf{r}_i = 3$, we get the well-known ideal gas equation of state:

$$3NkT = -\sum_i (-3)pV_i \quad \Rightarrow pV = nKT. \quad (11)$$

If force $\mathbf{F}$ is conservative, we can write $\mathbf{F}_i = -\nabla U(r)$:

$$2\langle K\rangle = \sum_i \langle \nabla U \cdot \mathbf{r}_i\rangle \equiv \sum_i \langle \frac{\partial U}{\partial r_i}\hat{\mathbf{r}}_i \cdot \mathbf{r}_i\rangle. \quad (12)$$

In this format, the virial theorem is thus written as:

$$2\langle K\rangle = \sum_i \langle r_i \frac{\partial U}{\partial r_i}\rangle. \quad (13)$$

We can further specify this by assuming a functional form for the potential, $U(r) = kr^{n+1}$:

$$2\langle K\rangle = \sum_i \langle r_i(n+1)U(r_i)\rangle = (n+1)\langle U\rangle. \tag{14}$$

For Hooke's law, for example, $n = 1$; for gravity, $n = -2$. Thus, for a system of particles in gravitational field:

$$2\langle K\rangle = -\langle U\rangle. \tag{15}$$

This is the virial theorem's most frequent form.

Now let us provide a slightly different derivation to provide context to the Q quantity. We start with Newton's law:

$$\sum_i \mathbf{F}_i = \sum_i m_i\ddot{\mathbf{r}}_i. \tag{16}$$

Dot-multiply this by $\mathbf{r}_i$:

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \sum_i m_i\mathbf{r}_i \cdot \ddot{\mathbf{r}}_i. \tag{17}$$

The left-hand side (called *virial*) we already recognize; for the right-hand side we employ a frequent trick; we note that:

$$(m_i r_i^2)^{\cdot\cdot} = (2m_i\mathbf{r}_i \cdot \dot{\mathbf{r}}_i)^{\cdot} = 2m_i(\dot{r}_i^2 + \mathbf{r}_i \cdot \ddot{\mathbf{r}}_\mathbf{i}). \tag{18}$$

Thus, solving for the last term on the right:

$$\sum_i m_i\mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = \sum_i \frac{1}{2}(m_i r_i^2)^{\cdot\cdot} - \sum_i m\dot{r_i}^2. \tag{19}$$

The first term on the right is directly related to our Q quantity; thus, we are talking about the second time derivative of the moment of inertia. For a system in a hydrostatic equilibrium, the time average of this term is again 0, and the rest is our virial theorem.

Another example: consider a collapsing spherical nebula. We want to calculate the total energy of the nebula.

We start with the virial theorem:

$$2\langle K\rangle = -\langle U\rangle. \tag{20}$$

As total energy $E = K + U$, it suffices to calculate U and we will obtain E:

$$E = K + U = -\frac{U}{2} + U = \frac{U}{2}. \tag{21}$$

So let us calculate U!

$$dU = -\frac{GM(r)dm}{r} = -\frac{GM(r)\rho(r)4\pi r^2 dr}{r}. \tag{22}$$

Assuming radial-only mass distribution, we can substitute $M(r) = \rho(r)V$ and integrate:

$$U = -G\frac{(4\pi)^2}{3}\int \rho^2(r)r^4 dr. \tag{23}$$

If we further assume that $\rho(r) \equiv \rho_0$, the final expression becomes:

$$U = -\frac{3GM^2}{5R} \quad \Rightarrow \quad E = -\frac{3GM^2}{10R}. \tag{24}$$

How much energy has been radiated away? For that we need to evaluate the difference $E_{\rm rad} = E_{\rm cloud} - E_{\rm star}$:

$$E_{\rm rad} = -\frac{3GM^2}{10R_{\rm cloud}} + \frac{3GM^2}{10R_{\rm star}} \approx \frac{3GM^2}{10R_{\rm star}}. \tag{25}$$

The approximation holds because $R_{\rm cloud} \gg R_{\rm star}$. Plugging in the numbers, we get $E_{\rm rad} \sim 10^{41}$J. If the solar luminosity were constant, this energy would be radiated away in:

$$t = \frac{E_{\rm rad}}{L_\odot} = \frac{10^{41}\,\rm J}{4 \times 10^{26}\rm W} = 10^7 \text{ years}. \tag{26}$$

How about the central temperature of the Sun?

$$K = \frac{3}{2}NkT = \frac{3}{2}\frac{MkT}{\mu m_H}, \quad U = -\frac{3GM^2}{5R}. \tag{27}$$

From the virial theorem it follows that:

$$2K + U = 0 \quad \Rightarrow \quad 3\frac{MkT}{\mu m_H} - \frac{3GM^2}{5R} = 0. \tag{28}$$

Solving for T:

$$T = \frac{GM\mu m_H}{5kR} = 2 \times 10^6\,\rm K. \tag{29}$$

That is ∼8 times too small, but that's because we have made some pretty stark approximations about $\rho(r)$.

When will a cloud collapse to a protostar? More of the same.

$$E = \frac{1}{2}U < 0; \quad 2K + U = 0; \quad K = \frac{3}{2}\frac{MkT}{\mu m_H}; \quad U = -\frac{3GM^2}{5R}. \tag{30}$$

Plugging it in, we obtain:

$$M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}. \tag{31}$$

This is known as Jeans' mass. If instead we solve for R, we refer to that as Jeans' length:

$$R_J = \left(\frac{15kT}{4\pi\rho G\mu m_H}\right)^{1/2}. \tag{32}$$

2 Homologous free-fall time

Assuming a spherical, equal density cloud isolated in space, the free-fall is determined by an isothermal collapse. We start with the equation of motion:

$$\ddot{r} = -\frac{GM_r}{r^2}, \tag{33}$$

where M_r is the mass enclosed within a spherical shell of radius r. Multiply by $\dot{r}$:

$$\dot{r}\ddot{r} = \left(\frac{1}{2}\dot{r}^2\right)^{\cdot} = -\frac{GM_r}{r^2}\dot{r}. \tag{34}$$

We can integrate this:

$$\frac{1}{2}\dot{r}^2 = -\frac{4\pi\rho_0 G r_0^3}{3}\int\frac{1}{r^2}dr = \frac{4\pi\rho_0 G r_0^3}{3r} + C, \tag{35}$$

where C is the integration constant. We get it by requiring that $\dot{r}|_{r=r_0} = 0$:

$$\frac{1}{2}\dot{r}^2 = \frac{4}{3}\pi\rho_0 G r_0^2 + C = 0 \quad \Rightarrow \quad C = -\frac{4}{3}\pi\rho_0 G r_0^2. \tag{36}$$

This yields the following expression:

$$\dot{r} = \pm\left[\frac{8}{3}\pi\rho_0 G r_0^2\left(\frac{r_0}{r} - 1\right)\right]^{1/2}. \tag{37}$$

Sign choice comes from the direction of collapse: it needs to be a negative sign. We integrate this again by introducing $x = r/r_0$ and denoting $\theta = (8\pi\rho_0 G/3)^{1/2}$:

$$r_0\dot{x} = -\theta r_0 \left(\frac{1}{x} - 1\right)^{1/2}. \tag{38}$$

Next we introduce a new variable ξ:

$$x = \cos^2\xi, \quad \dot{x} = -2\cos\xi\sin\xi\dot{\xi}. \tag{39}$$

From there:

$$\dot{x} = -2\cos\xi\sin\xi\dot{\xi} = \theta\left(\frac{1}{\cos^2\xi} - 1\right)^{1/2} \quad \Rightarrow \quad \cos^2\xi\dot{\xi} = \frac{\theta}{2}. \tag{40}$$

We integrate this again:

$$\int cos^2\xi d\xi = \frac{\theta}{2}\int dt = \frac{\theta}{2}t. \tag{41}$$

The left-hand side is integrated by noting the trig relationship for double angles:

$$\cos 2\xi = \cos^2\xi - \sin^2\xi = 2\cos^2\xi - 1 \quad \Rightarrow \quad \cos^2\xi = \frac{1}{2}(1 + \cos 2\xi). \tag{42}$$

The integral then becomes:

$$\int cos^2\xi d\xi = \frac{1}{2}\xi + \frac{1}{4}\int \cos 2\xi d(2\xi) = \frac{1}{2}\xi + \frac{1}{4}\sin 2\xi + D, \tag{43}$$

where D is, again, the integration constant. This time we use the constraint that $r|_{t=0} = r_0$ (so, in turn, $x|_{t=0} = 1$ and $\xi|_{t=0} = 0$):

$$D = 0 \quad \Rightarrow \quad \xi + \frac{1}{2}\sin 2\xi = \theta t. \tag{44}$$

Finally, we can evaluate $t|_{r\to 0}$, which is the homologous free-fall time:

$$t_{\text{ff}} = t|_{r=0} = \frac{\pi}{2\theta} = \left(\frac{3\pi}{32}\frac{1}{\rho_0 G}\right)^{1/2}. \tag{45}$$

3 Cloud fragmentation

Masses of large molecular clouds certainly exceed Jeans' mass. That would imply that such clouds can form supermassive stars, up to the initial mass of the cloud, but this clearly does not happen. Stars tend to form in multiples or open clusters, so something must be causing fragmentation.

As foray into the discussion of fragmentation, consider a collapsing cloud; its density certainly increases by several orders of magnitude during freefall. As temperature remains nearly constant, *Jeans mass must then decrease*. Thus, any initial inhomogeneities will cause individual sections of the cloud to satisfy the Jeans criterion independently and begin to collapse locally.

On the flip side, fragmentation must at some point stop, otherwise we would be creating a whole slew of small bodies. The main reason for this is that collapse goes from being isothermal to being adiabatic – otherwise stars would have 10-100 K. An isothermal collapse implies that all excess energy is efficiently radiated away, while an adiabatic collapse does not lose any energy and, in consequence, the gas *must* heat up.

3.1 Recap of adiabatic equation of state

First law of thermodynamics:

$$dU = dQ + dW = dQ - pdV. \tag{46}$$

Specific heat is the amount of heat required to raise the temperature of unit mass by dT:

$$c_p = \left(\frac{\partial Q}{\partial T}\right)_p, \quad c_v = \left(\frac{\partial Q}{\partial T}\right)_V. \tag{47}$$

Remember that U is *state quantity*, so it does not depend on any specific circumstance. That is why we can evaluate it, for example, at constant volume:

$$dU = \left(\frac{\partial Q}{\partial T}\right)_V dT \equiv c_v dT. \tag{48}$$

Because U is a state quantity, this holds generally, not just when V is constant. We can evaluate the first law of thermodynamics for a monoatomic ideal gas:

$$dU = \frac{3}{2}N_M k dT = \frac{3}{2}\frac{k}{\mu m_H}dT = \frac{3}{2}nRdT. \tag{49}$$

From there we can see the expression for R:

$$nR = \frac{k}{\mu m_H} = \frac{2}{3}c_V. \tag{50}$$

To find the relationship with c_P, we evaluate first law of thermodynamics at constant pressure:

$$dU = dQ - pdV = \left(\frac{\partial Q}{\partial T}\right)_p dT - p\left(\frac{\partial V}{\partial T}\right)_p dT. \tag{51}$$

Thus:

$$dU = c_p dT - nRdT = c_p dT - \frac{2}{3}c_V dT = (c_p - \frac{2}{3}c_V)dT = c_V dT. \tag{52}$$

From there it follows that:

$$c_V = c_p - \frac{2}{3}c_V \quad \Rightarrow \quad \frac{c_p}{c_V} = \frac{5}{3} \quad \text{for monoatomic gas.} \tag{53}$$

In general, we define $\gamma = c_p/c_V$ as the adiabatic constant. It tells us how rapidly will gas heat up or cool down during an adiabatic process (i.e., gas expanding/shrinking or compressing/rarefying). Larger value of γ means a larger temperature response to pressure/volume change. We can now derive the adiabatic equation(s) of state:

$$dU = dQ - pdV = -pdV = c_V dT. \tag{54}$$

Differentiating the ideal gas equation of state:

$$pdV + Vdp = nRdT \quad \text{(assuming constant } n\text{).} \tag{55}$$

Plugging the first equation into the second:

$$pdV + Vdp = nR\left(-\frac{p}{c_V}\right)dV. \tag{56}$$

Rearranging yields:

$$\left(1 + \frac{nR}{c_V}\right)\frac{dV}{V} = -\frac{dp}{p}. \tag{57}$$

Yet the expression in parentheses is precisely γ, so:

$$p \propto V^{\gamma} \ : \qquad \text{adiabatic equation of state.} \tag{58}$$

We can express any other thermodynamical quantity to get other forms of the adiabatic equation of state:

$$pdV + Vdp = nRdT \quad \Rightarrow \quad pdV = nRdT - Vdp, \tag{59}$$

and plugging into the first equation:

$$\gamma pdV = \gamma(nRdT - Vdp) = -Vdp \quad \Rightarrow \quad \gamma\frac{p}{T}dT = dp(\gamma - 1) \tag{60}$$

and, finally:

$$p \propto T^{\gamma/(\gamma-1)} \; ; \qquad \text{adiabatic equation of state.} \tag{61}$$

We can do the same with density:

$$p = \frac{\rho}{M}RT, \quad dp = \frac{R}{M}(\rho dT + Td\rho). \tag{62}$$

Plug these into the previous equation and we get:

$$T \propto \rho^{\gamma-1}. \tag{63}$$

3.2 Back to fragmentation

After that quick digression into thermodynamics, we can now go back to our question of fragmentation. How does Jeans' mass change if we use this adiabatic relationship?

$$3NkT - \frac{3}{5}\frac{GM^2}{R} = 0, \tag{64}$$

$$\frac{k}{\mu m_H}C\rho^{\gamma-1} = \frac{1}{5}\frac{GM}{R}. \tag{65}$$

Now we substitute R:

$$\frac{k}{\mu m_H}C\rho^{\gamma-1} = \frac{1}{5}GM\left(\frac{3M}{4\pi\rho}\right)^{-1/3}. \tag{66}$$

Solve for mass:

$$M_J^{2/3} = \frac{5kC}{G\mu m_H}\left(\frac{3}{4\pi}\right)^{1/3}\rho^{\gamma-1-1/3} \quad \Rightarrow \quad M_J \propto \rho^{(3\gamma-4)/3}. \tag{67}$$

If the cloud is all hydrogen, then $\gamma = 5/3$ and $M_J \propto \rho^{1/2}$. Thus, M_J *increases with increasing density for an adiabatic collapse*! There must be some minimum mass for fragments.

According to the virial theorem, energy must be liberated during collapse:

$$\Delta E_g = \frac{3}{10}\frac{GM_J^2}{R_J}, \quad L_{\rm ff} = \frac{\Delta E_g}{t_{\rm ff}} = \frac{12\sqrt{2}}{10} G^{3/2} \left(\frac{M_J}{R_J}\right)^{5/2}. \tag{68}$$

Emission is governed by radiation:

$$L_{\rm rad} = 4\pi R_J^2 \varepsilon \sigma T^4, \tag{69}$$

where we introduced ε as the efficiency parameter, which is between 0 (adiabatic) and 1 (blackbody). Equating the two yields:

$$M_J^{5/2} = \frac{4\pi}{G^{3/2}} R_J^{9/2} \varepsilon \sigma T^4. \tag{70}$$

Now we eliminate the radius:

$$M_J^{5/2} = \frac{4\pi}{G^{3/2}} \left(\frac{3M_J}{4\pi\rho}\right)^{3/2} T^4 \tag{71}$$

and express density with Jeans' mass:

$$\left(\frac{3}{4\pi\rho}\right)^{1/2} = M_J \left(\frac{5kT}{G\mu m_H}\right)^{-3/2}, \tag{72}$$

yielding:

$$M_J = 0.03 M_\odot \frac{T^{1/4}}{\varepsilon^{1/2}\mu^{9/4}}. \tag{73}$$

For typical values of T, ε and μ, we get $\sim$0.1-0.5 $M_\odot$.

4 Freefall collapse beyond analytical models

We have of course neglected a lot of important physical contributions that could not be added to the analytical model. For example,

- initial velocity of the cloud's outer layers;

- radiation transport through the cloud;
- vaporization of dust grains;
- dissociation of molecules, ionization of atoms;
- rotation/angular momentum;
- magnetic fields; etc.

To get around this, we need to solve MHD equations numerically.

4.1 Phenomenological description

Consider a supercritical spherical cloud of $\sim 1 M_\odot$. The evolution pathway consists of the following steps:

- initial freefall is nearly isothermal because light can escape from the cloud;
- because of the mass buildup in the center, density increases, causing faster core contraction;
- when density reaches about $10^{-10}\,\mathrm{kg/m^3}$, the cloud becomes optically thick due to dust;
- this makes contraction more adiabatic; increased pressure slows down core contraction;
- the central region establishes a quasi-hydrostatic equilibrium with a $\sim 5\,R_\odot$ radius. That is the protostar;
- around the protostar the material is still in freefall. Once it falls onto the core, it produces a supersonic shock wave that loses its kinetic energy and produces heat that powers core luminosity;
- when the temperature reaches $\sim 1000\,\mathrm{K}$, dust vaporizes and opacity drops. But since luminosity stays high, that necessarily implies a rising temperature;

- as the material keeps falling in and the temperature keeps rising, once it reaches $\sim$2000 K, molecular hydrogen dissociates into atoms. This reduces the pressure gradient, destabilizing the core and the second collapse occurs. The radius is reduced to $\sim$130% of the present size, where hydrostatic equilibrium is again established;

- as the envelope continues to accrete material, deuterium starts to burn:

$$ {}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He} + \gamma \tag{74} $$

  This produces $\sim$60% of the luminosity;

- as both the infalling mass and deuterium are limited, the luminosity eventually decreases. This is called deuterium burnout. The protostar cools slightly.

This whole sequence takes $\sim t_{\rm ff}$ time. Fig. 1 depicts the numerically computed evolutionary tracks for the described scenario. Once we have a quasi-static protostar, the rate of evolution is dictated by the stars's ability to thermally adjust to contraction. We have estimated this before already: the main agent is gravity, so we recall that potential energy is:

$$ U = -\frac{3}{5}\frac{GM^2}{R}. \tag{75} $$

Important aspect: ΔU is *negative* for $r_2 < r_1$, and because $E_{\rm tot}$ needs to be conserved, the negative excess is compensated by a positive change:

$$ \Delta U = -\frac{3}{5}GM^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) < 0 \quad \Rightarrow \quad \Delta E > 0. \tag{76} $$

Thus, this will produce heat, half of which will be radiated away (half because of the virial theorem):

$$ \Delta E_{\rm tot} = \frac{3}{5}\frac{GM_\odot^2}{R_\odot} \quad \Rightarrow \quad E_{\rm rad} = \frac{3}{10}\frac{GM_\odot^2}{R_\odot}. \tag{77} $$

The timescale, $t_{\rm KH} \sim E_{\rm rad}/L_\odot$, is called the Kelvin-Helmholtz timescale and is $\sim 10^7$ years for the Sun.

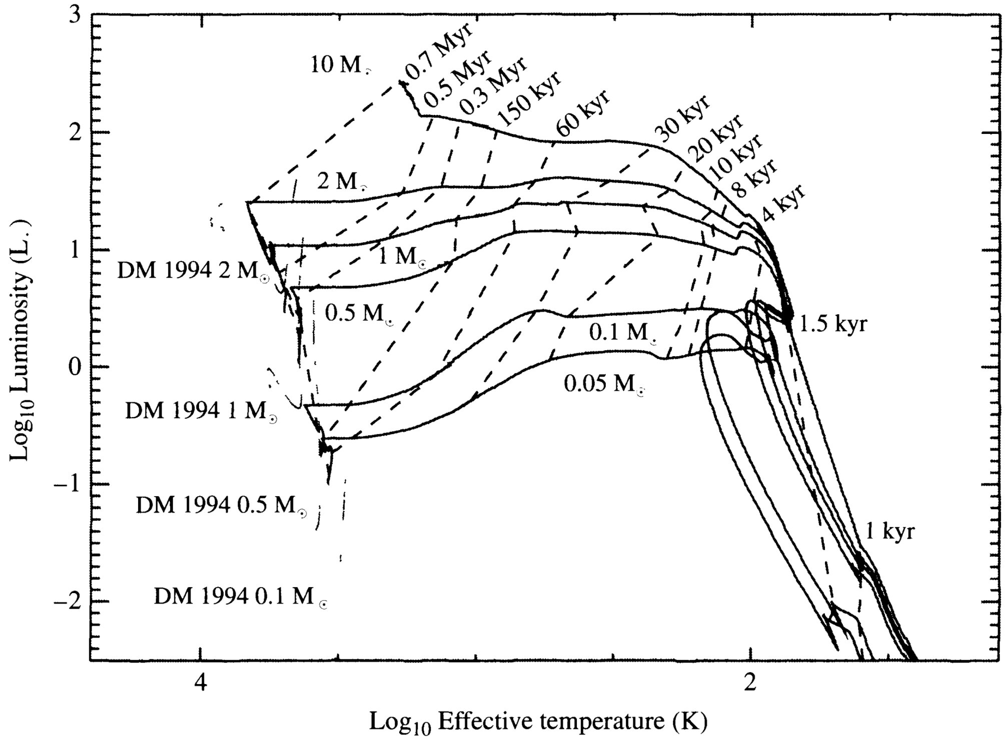

Figure 1: Evolutionary tracks for the cloud collapse into a protostar.

5 Stellar structure

Let us start the discussion of stellar structure with underlying assumptions.

- stars are considered isolated, their evolution only depends on their intrinsic properties;
- stars form from a homogeneous cloud;
- stars are spherically symmetrical, which is akin to saying that we neglect rotation and magnetic fields.

Given the assumed spherical symmetry, all interior physical quantities depend only on r:

$$p = p(r), \quad T = T(r), \quad \rho = \rho(r), \quad \ldots \tag{78}$$

In an evolving star, we also have time as an independent quantity. When we evaluate derivatives, we do that with respect to r and t:

$$dX = \left(\frac{\partial X}{\partial r}\right)_{t=\text{const.}} dr + \left(\frac{\partial X}{\partial t}\right)_{r=\text{const.}} dt. \tag{79}$$

Consider, for example, a spherical mass shell at distance r from the center and thickness dr; mass differential becomes:

$$dm(r,t) = \frac{\partial m}{\partial r} dr + \frac{\partial m}{\partial t} dt = 4\pi r^2 \rho dr - 4\pi r^2 \rho v dt, \tag{80}$$

where v is the radial velocity of mass inside the shell, and – in general – both $\rho = \rho(r,t)$ and $v = v(r,t)$. The first term on the left is the first fundamental equation of stellar structure:

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho(r,t). \tag{81}$$

The second term describes mass loss/gain by motion; it is generally negligible in stellar interiors but can be important in the envelopes (stellar wind, accretion).

Since mass increases monotonically outward, we can also use $M(r)$ instead of r:

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r}\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho(r,t)}\frac{\partial}{\partial r}. \tag{82}$$

Thus, the first equation of stellar structure can also be written as:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}. \tag{83}$$

Note that, if $r \neq r(t)$, this becomes an ordinary differential equation.

5.1 Poisson equation for gravity

Say we wanted to compute the gravitational field in point $\mathbf{r}$ due to all masses present in space S. We need to integrate:

$$\mathbf{g} = -G \int_{\mathrm{S}} \frac{\mathbf{r} - \mathbf{s}}{||\mathbf{r} - \mathbf{s}||^3}\, dM = -G \int_{\mathrm{S}} \frac{\mathbf{r} - \mathbf{s}}{||\mathbf{r} - \mathbf{s}||^3} \rho \, d\mathbf{s}, \tag{84}$$

where $\mathbf{s}$ should be taken as $ds_x ds_y ds_z$, i.e. a volume element. Evaluate a divergence of both sides:

$$\nabla \cdot \mathbf{g} = -G \int_S \nabla \cdot \left(\frac{\mathbf{r} - \mathbf{s}}{||\mathbf{r} - \mathbf{s}||^3} \right) d\mathbf{s}. \tag{85}$$

Let us take a closer look at the divergence under the integral:

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}. \tag{86}$$

This results in:

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = \frac{1}{r^6} \left[3r^{3/2} - 3r^{1/2}(x^2 + y^2 + z^2) \right] = 0 \tag{87}$$

for any $\mathbf{r} \neq \mathbf{0}$. How about when $\mathbf{r} = \mathbf{0}$? We apply Gauss's law:

$$\int_S \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) d\mathbf{S} = \int_{\partial S} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{A} = \int_{\partial S} \frac{1}{r^3} \mathbf{r} \cdot \hat{\mathbf{r}} \, dA = \int_{\partial S} \frac{dA}{r^2}, \tag{88}$$

where $dA = r^2 \sin\theta d\theta d\phi$ is a spherical surface element. Substituting that in:

$$\int_{\partial S} \nabla \phi \cdot d\mathbf{A} = \int_0^{2\pi} \int_0^{\pi} \nabla \phi \cdot \hat{\mathbf{r}} \, r^2 \sin\theta d\theta d\phi. \tag{89}$$

We can immediately integrate along ϕ, yielding 2π; we also note that $\phi \propto 1/r$, and $\nabla \phi = -\mathbf{r}/r^3$:

$$\int_0^{2\pi} \int_0^{\pi} \nabla \left(\frac{1}{r} \right) r^2 \sin\theta d\theta d\phi = -2\pi \int_0^{\pi} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \sin\theta d\theta = 4\pi. \tag{90}$$

We can now write the auxiliary divergence that encompasses both $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \neq \mathbf{0}$:

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = 4\pi \delta(\mathbf{r}), \tag{91}$$

where $\delta(\mathbf{r})$ is the Dirac delta function. Use this in the original integral:

$$\nabla \cdot \mathbf{g} \equiv -\nabla^2 \Phi = G \int_S \nabla \cdot \left(\frac{\mathbf{r} - \mathbf{s}}{||\mathbf{r} - \mathbf{s}||^3} \right) \rho \, d\mathbf{s} = -4\pi G \rho. \tag{92}$$

This is the Poisson equation for gravity, and Φ is the potential that solves it.

5.2 Spherically symmetrical case

If $g = g(r)$, this reduces to the familiar equation; we start from the Poisson equation:

$$\nabla \cdot \mathbf{g} = -4\pi\rho G. \tag{93}$$

Divergence in spherical units is:

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial(r^2 f(r))}{\partial r} + \text{angular terms}, \tag{94}$$

so:

$$\nabla \cdot \mathbf{g} = \frac{1}{r^2}\frac{\partial(r^2 g(r))}{\partial r} = -4\pi\rho G. \tag{95}$$

Rearrange and integrate:

$$r^2 g(r) = -4\pi\rho G \int r^2 dr = -\frac{4}{3}\pi\rho G r^3 \equiv GM \quad \Rightarrow \quad g(r) = -\frac{GM}{r^2}. \tag{96}$$

5.3 Equations of motion and hydrostatic equilibrium

6 Polytropes and the Lane-Emden equation

6.1 Interpreting the solutions

In the previous section we introduced a scaling parameter α as:

$$\alpha = \left(\frac{4\pi G}{K(n+1)}\rho_c^{(n-1)/n}\right)^{1/2}. \tag{97}$$

We can now "unscale" ξ to get the radius of a polytrope for the given value of n:

$$r = \frac{\xi}{\alpha} = \xi\left(\frac{4\pi G}{K(n+1)}\rho_c^{(n-1)/n}\right)^{-1/2}; \tag{98}$$

if this is evaluated at $\xi_n \equiv \xi|_{y=0}$, we get the polytrope radius:

$$R_n = \xi_n\left(\frac{K(n+1)}{4\pi G}\right)^{1/2}\rho_c^{(1-n)/2n}. \tag{99}$$

Similarly, we can obtain the mass of the polytrope. Start from the first equation of stellar structure:

$$m(r) = \int 4\pi r^2 \rho dr = \int 4\pi \frac{\xi^2}{\alpha^2}\rho_c y^n \frac{d\xi}{\alpha} = \frac{4\pi\rho_c}{\alpha^3}\int \xi^2 y^n d\xi. \tag{100}$$

The integrand looks just like the right-hand side of the Lane-Emden equation (to the negative sign), so we substitute it in:

$$m(\xi) = -\frac{4\pi\rho_c}{\alpha^3}\int \frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right)d\xi. \tag{101}$$

Thus, this simplifies to the final form:

$$m(\xi) = -\frac{4\pi}{\alpha^3}\xi^2\frac{dy}{d\xi}. \tag{102}$$

It proves useful to introduce an auxiliary function $\theta(\xi)$:

$$\theta(\xi) = -\xi^2\frac{dy}{d\xi} \quad \Rightarrow \quad m(\xi) = \frac{4\pi}{\alpha^3}\theta(\xi). \tag{103}$$

If we substitute the expression for α and evaluate it at $\xi = \xi_n$, we obtain the polytrope mass:

$$M_n = (4\pi)^{-1/2}\left[\frac{K(n+1)}{G}\right]^{3/2}\rho_c^{(3-n)/2n}\theta(\xi_n). \tag{104}$$

Finally, we can get a useful relationship between M_n and R_n if we eliminate ρ_c from both expressions. After some manipulation:

$$K = C_n R^{(3-n)/n} M^{(n-1)/n}, \quad \text{where} \tag{105}$$

$$C_n = \frac{(4\pi)^{1/n}}{n+1}G^{(3-n)/2n}\xi^{(n-3)/n}\theta^{(1-n)/n}(\xi_n). \tag{106}$$

This helps us appreciate two special cases: $n = 1$ and $n = 3$. In case of $n = 1$, radius depends exclusively on K and is independent of mass; for $n = 3$, mass depends exclusively on K and is independent of radius. To put it differently, for a given K there is a single value of R (for $n = 1$) or M (for $n = 3$) that can establish a hydrostatic equilibrium. We can also relate central density to the average density:

$$\langle\rho\rangle = \frac{3M}{4\pi R^3} \tag{107}$$

by plugging in the expressions for M_n and R_n:

$$\langle\rho\rangle = 3\xi_n^{-3}\theta(\xi_n)\rho_c. \tag{108}$$

Their ratio, $\langle\rho\rangle/\rho_c$, is the degree of central concentration of a polytrope. It only depends on the polytropic index n.

6.2 Specific solutions

As mentioned above, there are three values of n that have analytical solutions: $n = 0$, $n = 1$, and $n = 5$. Let us explore those solutions.

6.2.1 The $n = 0$ solution

This is the simplest solution as it represents a homogeneous gas sphere with constant density ρ_c.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right) = -1. \tag{109}$$

This equation is fully integrable and it yields a solution:

$$y = 1 - \frac{\xi^2}{6}. \tag{110}$$

The value of ξ at which y assumes 0 gives us the scaled radius of the star:

$$0 = 1 - \frac{\xi_n^2}{6} \quad \Rightarrow \quad \xi_n = \sqrt{6}. \tag{111}$$

6.2.2 The $n = 1$ solution

This time we are solving the following equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right) = -y. \tag{112}$$

This differential equation can be solved by series expansion. We set:

$$y(\xi) = \sum_k a_k\xi^k; \quad y'(\xi) = \sum_k k a_k\xi^{k-1}; \quad y''(\xi) = \sum_k k(k-1)a_k\xi^{k-2}. \tag{113}$$

Plugging this into the Lane-Emden equation yields the following relationship:

$$\sum_k [k(k-1) + 2k]a_k\xi^{k-2} = -\sum_k a_k\xi^k. \tag{114}$$

Introduce $l = k - 2$:

$$\sum_l (l+1)(l+2) + 2(l+2)a_{l+2}\xi^l = -\sum_l a_l\xi^l, \tag{115}$$

where we renamed the dummy variable on the right-hand side from k to l. Now we can figure out the recursion:

$$a_{l+2} = -\frac{a_l}{(l+2)(l+3)}. \tag{116}$$

The last thing to do is to write out the series:

$$y = a_0\xi^0 + a_1\xi^1 + a_2\xi^2 + \dots \tag{117}$$

$$y = a_0 + a_1\xi - \frac{a_0}{4\times 5}\xi^2 - \frac{a_1}{5\times 6}\xi^3 \dots \tag{118}$$

Now we group the terms with a_0 and a_1 and see if it is anything recognizable:

$$y = a_0\left[1 - \frac{\xi^2}{4\times 5} + \frac{\xi^4}{6\times 7} - \frac{\xi^6}{8\times 9} + \dots\right] +$$

$$+a_1\left[\xi - \frac{\xi^3}{5\times 6} + \frac{\xi^5}{7\times 8} - \frac{\xi^7}{9\times 10} + \dots\right]. \tag{119}$$

These two series are Bessel series; from the boundary condition $y(\xi = 0) = 1$ we see that $a_1 = 0$ and $a_0 \neq 0$. Thus, the solution is:

$$y = a_0\sum_k \mathrm{mod(k, 2)}(-1)^k\frac{\xi^k}{(k+2)(k+3)} = a_0 j_0(\xi) = \frac{\sin\xi}{\xi}. \tag{120}$$

To find the scaled radius, we set this to 0:

$$0 = \frac{\sin\xi_n}{\xi_n} \Rightarrow \xi_n = \pi. \tag{121}$$

6.2.3 The $n = 5$ solution

The last analytical solution can be obtained by proposing a trial solution:

$$y(\xi) = (1 + A\xi^2)^{1/2}. \tag{122}$$

Note: you can arrive to the same answer by first introducing a substitution $u = \xi y$, then $v = u^{-2}$ and then integrating. But if we stick with the trial solution:

$$\frac{dy}{d\xi} = a\xi(1 + A\xi^2)^{-3/2}, \tag{123}$$

which, after plugging into the Lane-Emden equation, leads to:

$$-3A\left[(1+A\xi^2)-A\xi^2\right]=-1 \quad \Rightarrow \quad A=\frac{1}{3} \tag{124}$$

and the final solution:

$$y(\xi)=\left(1+\frac{\xi^2}{3}\right)^{-1/2}. \tag{125}$$

As before, equating this to 0 yields the radius, which in this case is infinite:

$$0=\frac{1}{\left(1+\frac{\xi_n^2}{3}\right)^{1/2}}\approx\frac{1}{1+\frac{\sqrt{3}}{3}\xi_n} \quad \Rightarrow \quad \xi_n\to\infty. \tag{126}$$