

Compartment models

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Chapter 1

Compartment models

Background: It is important to master the ability to develop models. Modeling of dynamical systems plays a very important role in applied science, and compartment models are among the most important tools used for analyzing dynamical systems.

Aim: The aim with this chapter is to learn more about development of compartment models.

1.1 Introduction

Compartment models are often used to describe transport of material in biological systems. A compartment model contains a number of compartments, each containing well mixed material. Compartments exchange material with each other following certain rules. Figure 1.1 shows a sketch of such a system. In this figure, compartments are represented by boxes and the connections between the compartments are represented by arrows. Every compartment (that is every box) has a number of connections leading to the box (inflows) and a number of arrows leading from the box (outflows). Material can either flow from one compartment to another, it can be added from the outside through a source, or it can be removed through a drain or a sink. Think of a bathtub, where water (the well-mixed material) is added through the faucet and leaves through the drain. In the example above, the material was water, but it can be used in a more abstract way. Generally, the material represents the amount of something that we wish to account for. To account for the material the model must fulfill some conservation law. In the bathtub example, we could develop a model based on conservation of mass. Most compartment models (as the one shown in Figure 1.1) have more than one compartment and equations for such a model are obtained by describing a conservation law for each compartment. Conservation laws

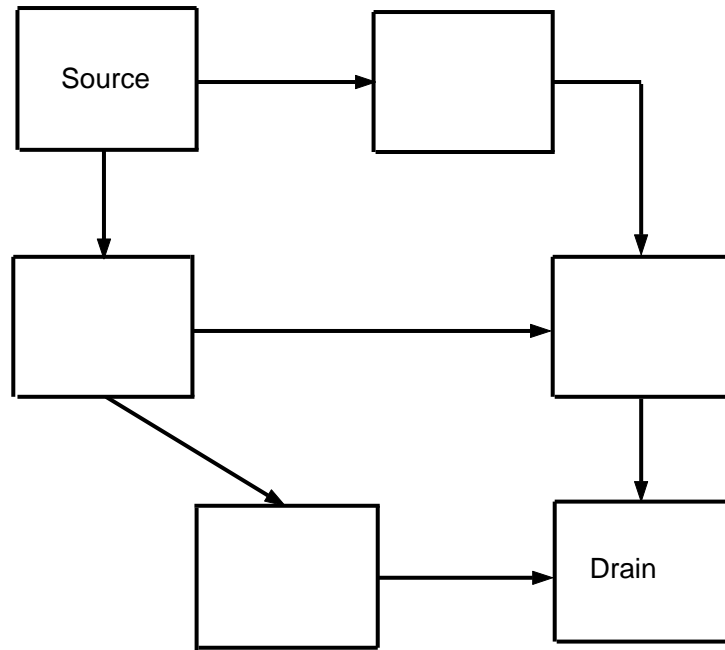


Figure 1.1: Sketch of a compartment model. Material can be stored in the boxes and transported between boxes following arrows.

state that the difference between what flows in and what flows out amounts to how much will be stored in the compartment.

A compartment model could also represent an ecological system where the material could be energy, the compartments could represent different species of animals and plants, and the flow between compartments could account for uptake and loss of food (or energy). In this case we would base the equations on laws describing conservation of energy. Compartment models also arise in physiology, where the material could be oxygen that is transported with the blood between different organs (compartments) in the body.

It should be emphasized, that one cannot think of compartments and the flows in and out of compartments as individual components where each part can be described independently of each other. Both the in- and the outflow from any compartment may depend on the volume inside the compartment. Similarly, the inflow into a compartment may depend on the outflow from another compartment. In other words, it is important to think of the system as a whole, where the parameter representing the material in the compartment (the state-variable) can depend on what flows in and what flows out. In addition, since what flows into one compartment typically flows out of another compartment, the state-variables depend on each other and on the state of the system as a whole. The important point to remember is, that it is the person modeling the system who chooses how the

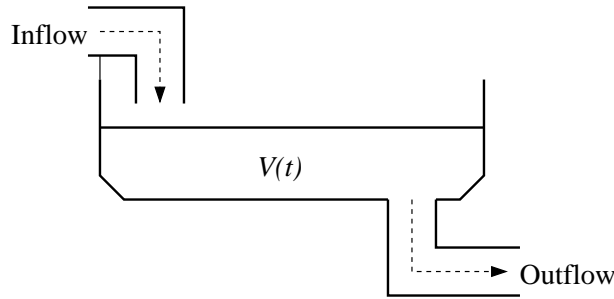


Figure 1.2: A bathtub with an external source (the faucet) and sink (the drain).

model parameters and variables, in a consistent way, depend on each other.

Before we start with a more detailed description of compartment models and their abilities and limits, we will discuss a few examples and demonstrate how conservation of mass can be used.

1.1.1 Example 1: A Bathtub

Figure 1.2 shows a bathtub, that at the start of the investigation (at $t = 0$) contains $V_0 = 100$ l (liter) of water. When the faucet is opened water flows into the bathtub with a velocity of 5 l/min. If the drain is opened, an additional 4 l/min will flow out of the bathtub. Now, let us derive an equation that can be used to describe the volume $V(t)$ l of water in the bathtub as a function of time. The material that is transported in and out of the bathtub is water, and it is our job to determine how the volume of water changes with time, i.e. we want to determine the function $V(t)$.

The function $V(t)$ can be determined using simple calculations, however, we will use this example to establish a more general method, that also works for more complicated examples, such as example 2 discussed in the next section. It is our aim to develop an equation, that describes how the volume changes during a small period of time (from t to $t + \Delta t$). The change in volume during the time increment Δt is given by $V(t + \Delta t) - V(t)$ (this can be both positive and negative). Since water is an incompressible fluid, conservation of mass states that the change in volume equals the difference between what flows in and what flows out from the bathtub. Assume that both the faucet and the drain are open. Then, the flow into the bathtub is 5 l/min and the flow out of the bathtub is 4 l/min. In other words

$$\begin{aligned} V(t + \Delta t) - V(t) &= 5 \text{ l/min} \cdot \Delta t \text{ min} - 4 \text{ l/min} \cdot \Delta t \text{ min} \\ &= 1 \text{ l/min} \cdot \Delta t \text{ min.} \end{aligned}$$

If both sides of the equation are divided by Δt we get the following difference equation:

$$\frac{(V(t) + \Delta t) - V(t)}{\Delta t} = 1 \text{ l/min.}$$

If we let $\Delta t \rightarrow 0$ we recover the derivative of $V(t)$

$$V'(t) = 1 \text{ l/min.}$$

From this equation it becomes clear, that our method does not provide a direct way for determining the volume $V(t)$, but instead we get the derivative $V'(t)$. However, using integration we can solve this equation for $V(t)$. For any constant C , the solution is given by

$$V(t) = 1 \text{ l/min} \cdot t \text{ min} + C \text{ l.} \quad (1.1)$$

All functions of the above type are candidates for the complete solution. To find the exact solution we need to have one more piece of information. For, example, if we know that $V(0) = 100 \text{ l}$ we can find C as

$$V(0) = 100 \text{ l} = 1 \text{ l/min} \cdot 0 \text{ min} + C \text{ l} \quad \Leftrightarrow \quad C = 100 \text{ l.}$$

Inserting this value for C into the general solution (1.1) we can find the specific form of $V(t)$. For this example, the specific solution is given by

$$V(t) = 1 \text{ l/min} \cdot t \text{ min} + 100 \text{ l.}$$

If both the inlet and the outlet are open, and we start with 100 l of water, then every 1 min we add 1 l of water to the bathtub.

In this example, we could have gotten to the same conclusion without introducing this complex notation, but in the next example, we will see that our notation is necessary to solve the problem. Before we move on to the next example, let us reflect upon what we have learned so far. We have illustrated our results using the diagram shown in Figure 1.3. We have used a compartment with the volume of water representing the state variable $V(t)$. Water flows into the bathtub at a rate of 5 l/min (this is represented by an arrow pointed into the compartment, V_{inflow}) and water is flowing out of the bathtub with a rate of 4 l/min (this is represented by an arrow pointing away from the compartment, $V_{outflow}$). Note, that the arrows indicate what way the water is flowing.

1.2 Example 2: Salt Dissolved in Water

The second example describes water flowing through a tank. Our focus will not be on the water, but on how much salt (NaCl) that is dissolved in the water. At the start of our

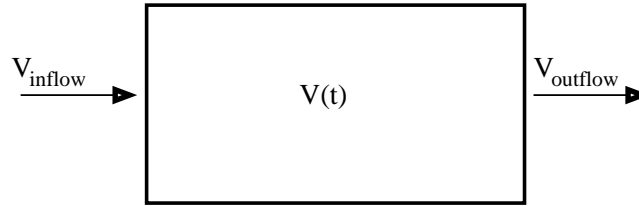


Figure 1.3: Compartment illustrating in- and outflow of water.

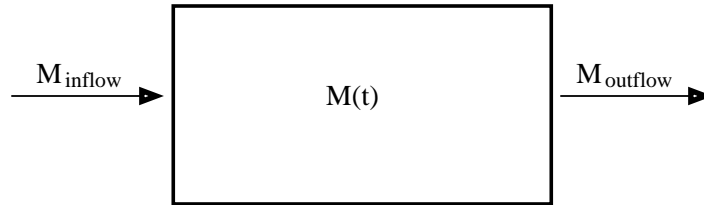


Figure 1.4: Compartment illustrating in- and outflow of salt-water. Note, in this example the compartment illustrate the amount of salt stored.

investigation (at $t = 0$) our tank has 0.8 kg of salt which is dissolved in $V_0 = 300$ l of water. We now add salt-water to the tank at a velocity of 5 l/min. This water contains 0.006 kg of salt per liter of water. The well-mixed salt-water is drained from the tank with the same velocity of 5 l/min, i.e. the volume of water in the tank does not change. Our goal is to calculate the amount of salt $M(t)$ (mass in kg) in the tank at time t . To do so, we choose the amount of salt to be the quantity that we want to keep track of. Hence, the compartment will no longer represent the amount of water, but an abstraction, the amount of salt in the water. What flows in and out of the compartment is an amount of salt per unit of time M_{inflow} and M_{outflow} , this is illustrated in Figure 1.4.

Again, the first step towards determining $M(t)$ is to derive an equation for mass conservation during a period of time (from t to $t + \Delta t$). During this period of time the amount of salt flowing into the compartment can be computed as

$$M_{\text{inflow}} = 5 \text{ l/min} \cdot \Delta t \text{ min} \cdot 0.006 \text{ kg/l} = 0.03 \text{ kg/min} \cdot \Delta t \text{ min}.$$

At time t , the water flowing out has a salt concentration of $M(t)/V_0$ (remember the volume in the container remains constant, since water flows in and out with the same velocity). Hence, the amount of salt flowing out can be computed by

$$M_{\text{outflow}} = 5 \text{ l/min} \cdot \Delta t \text{ min} \cdot \frac{M(t)}{V_0} \text{ kg/l}.$$

The difference equation, obtained by computing the difference between what flows in and

what flows out, divided by Δt is given by

$$\begin{aligned}\frac{M(t + \Delta t) - M(t)}{\Delta t} &= 0.03 \text{ kg/min} - \frac{5M(t)}{V_0} \text{ kg/min} \\ &= 0.03 \text{ kg/min} - \frac{M(t)}{60} \text{ kg/min}.\end{aligned}$$

The latter is obtained by using that the volume $V_0 = 300$ l. The derivative $M'(t)$ can be obtained by letting $\Delta t \rightarrow 0$

$$M'(t) = 0.03 \text{ kg/min} - \frac{M(t)}{60} \text{ kg/min}. \quad (1.2)$$

This is a linear differential equation, where $M'(t)$ is a function of $M(t)$. A function $M(t)$ that fulfills the above equation is the solution of the differential equation. This differential equation can be solved using separation of variables. The solution is given by

$$M(t) = 1.8 \text{ kg} + C \text{ kg} e^{-t/60}.$$

To prove that this solution fulfills the above differential equation, we differentiate $M(t)$ and compare the derivative with the right hand side of (1.2). To determine the constant C we need to know an initial value for $M(t)$. Let $M(0) = 0.8$ kg., then

$$M(0) = 0.8 \text{ kg} = 1.8 \text{ kg} + C \text{ kg} e^{-0/60} \Leftrightarrow C = -1.0 \text{ kg}.$$

Hence,

$$M(t) = (1.8 - e^{-t/60}) \text{ kg}. \quad (1.3)$$

Differentiating the solution gives:

$$M'(t) = \frac{1 \text{ kg}}{60 \text{ min}} e^{-t/60}$$

Inserting the above derivative into the differential equation (1.2) gives:

$$M'(t) = 0.03 \text{ kg/min} - \frac{1.8 \text{ kg} e^{-t/60}}{60 \text{ min}} = \frac{e^{-t/60}}{60} \text{ kg/min}.$$

The result (1.3) can be interpreted as follows: Water is poured into the tank with a concentration of 0.006 kg/l. Hence, with time the water in the tank will obtain exactly that concentration. Since the tank contains 300 l of water, we will have $300 \text{ l} \times 0.006 \text{ kg/l} = 1.8 \text{ kg}$ salt. This works well with the solution in (1.3) because if we let $t \rightarrow \infty$ the exponential term will go towards 0, written formally we get $\lim_{t \rightarrow \infty} e^{-t/60} = 0$. The solution for $M(t)$ is shown in Figure 1.5.

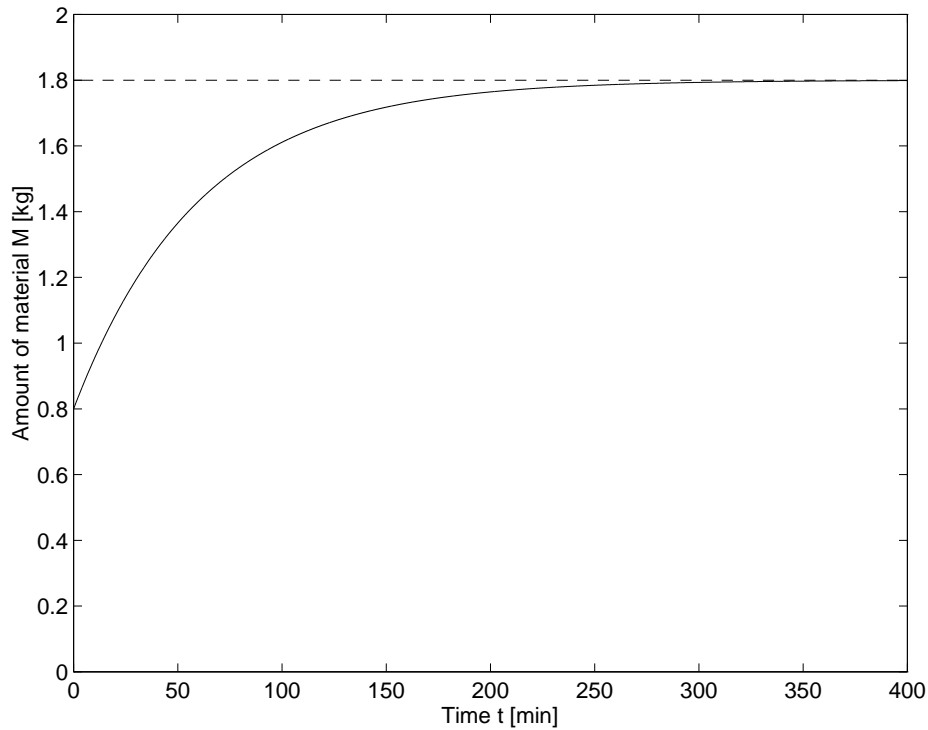


Figure 1.5: Graph representing the solution $M(t)$ to the differential equation in (1.6).

1.3 General Terms

Based on the two examples discussed above we can introduce some general concepts used to formulate a compartment model. Note that even though both models discussed above only had one compartment, a general compartment model will have multiple compartments. Since the material in each compartment must fulfil a conservation law, the general concepts arising from compartments can all be described by studying a single compartment. Models with more compartments will be discussed in Chapter 2.

We will use a model with one compartment to develop a number of general terms. The transport in and out of the compartment is characterized by the **velocity** of the inflow and the outflow, respectively. Velocity is measured in an amount of material per unit of time. Consider a compartment that is empty at $t = 0$. If the flow into the empty compartment is I [kg/min] and the flow out of the compartment is O [kg/min], then it will take $T = M/I$ [kg]/[kg/min] = [min] time units to fill the compartment (if $O = 0$), where M is the amount of material that can be stored in the full compartment. Similarly, if we consider a full compartment, it will take $T = M/O$ [kg] / [kg/min] = [min] time units to empty the compartment (if $I = 0$). If we consider a compartment with constant in- and outflow, then $M/I = M/O$. The time $T = M/I (= M/O)$ is often called the **turn-over time**, while the

velocity of the material flowing into the compartment is called the **turn-over rate**.

In the salt-water example discussed above, we computed the amount of salt in the compartment. A closer study of this model reveals that the outflow can be described by

$$O = aM$$

where a is a real constant. Computing outflow as described above is very common, both for examples as the one described above, but more generally in situations where every particle leaves the compartment with a given probability, independent of how long the particle has been stored inside the compartment. For example, if the compartment contains living animals, and we expect that the animals die with a given probability, then the rate of what flows out from the compartment have the form listed above. The factor of proportionality a is called a **rate**. In this example, this factor is the death-rate. The rate a has an interesting interpretation: In the stationary situation, i.e. if the inflow I equals the outflow O , then $a = O/M$. Hence, $a = I/M$ or using previous information we get $a = 1/T$.

In general, the change of mass in a compartment during a time interval Δt can be written as

$$M(t + \Delta t) - M(t) = I\Delta t - O\Delta t$$

where I is the velocity of the incoming fluid and O is the velocity of the fluid that is leaving. as before, a differential equation is obtained if we divide by Δt and take the limit as $\Delta t \rightarrow 0$, we get:

$$M'(t) = I - O$$

This equation states that the change of material with time is given by the difference between the inflow and the outflow. In the following we use this approach to calculate concentration of salt in a compartment. It should be noted that this approach only works if the velocities for the in- and out-flowing material are known. We use these velocities to determine the growth of the concentration in the compartment.

In the following section, we will discuss how a number of situations can be included into this compartment strategy. Both sections discuss **linear compartment models**, i.e. models where the rates are constant. However, in later sections we discuss models where the rates are dependent of the amount of material in the various compartments.

1.4 Balance of Concentrations

In this example we will use a compartment model to determine the balance between concentrations (this is similar to the mass-balances, however, instead of balancing masses we balance concentrations).

Again, we depart from the salt-water model model. Denote the amount of salt (in kg) that is dissolved in a volume V_0 of water by M . In the example above, we showed that the change of the amount of salt in the compartment can be computed as

$$M'(t) = 0.030 \text{ kg/min} - \frac{1}{60 \text{ min}} M(t)$$

(this was shown in equation (1.2)). In example 2, the differential equation for the mass $M'(t)$ was determined based on an inflow $I = 5$ (l/min) and the concentration of salt in the inflow $c_i = 0.006$ (kg/l). Using the symbols I and c_i , equation (1.2) can be written as

$$M'(t) = I c_i - \frac{O}{V_0} M \quad (1.4)$$

Note, in the example discussed above $I = O$ and the equation can be rewritten as

$$M'(t) = I c_i - \frac{I}{V_0} M \quad (1.5)$$

From this notation, it becomes clear that we can find the mass $M(t)$ by solving a linear first order differential equation. In the previous section we showed, that the solution to the salt-water equation is given by

$$M(t) = 1.8 \text{ kg} - 1.0 \text{ kg} e^{-t/60}$$

In this example we are interested in determining the concentration of salt, not the mass. To do so, we need to establish a fundamental law for the concentration. By definition concentration can be computed as mass divided by the volume, i.e. $c_i = M(t)/V_0$. Since the volume V_0 is constant differentiation yield $c'_i(t) = M'(t)/V_0$. Inserting (1.4) for $M'(t)$ gives:

$$c'(t) = \frac{I}{V_0} c_i - \frac{I}{V_0} c$$

If we let the quantity $a = I/V_0$ this equation can be rewritten as

$$c'(t) = a(c_i - c)$$

This is again a linear differential equation of first order. Studying the equation in terms of changes in concentration seems realistic. If the concentration in the water flowing into the compartment is bigger than the concentration of the material inside the compartment, then the concentration grows, i.e. $c_i > c$ and $c'(t) > 0$. If instead $c_i < c$ then the concentration of salt in the compartment will fall.

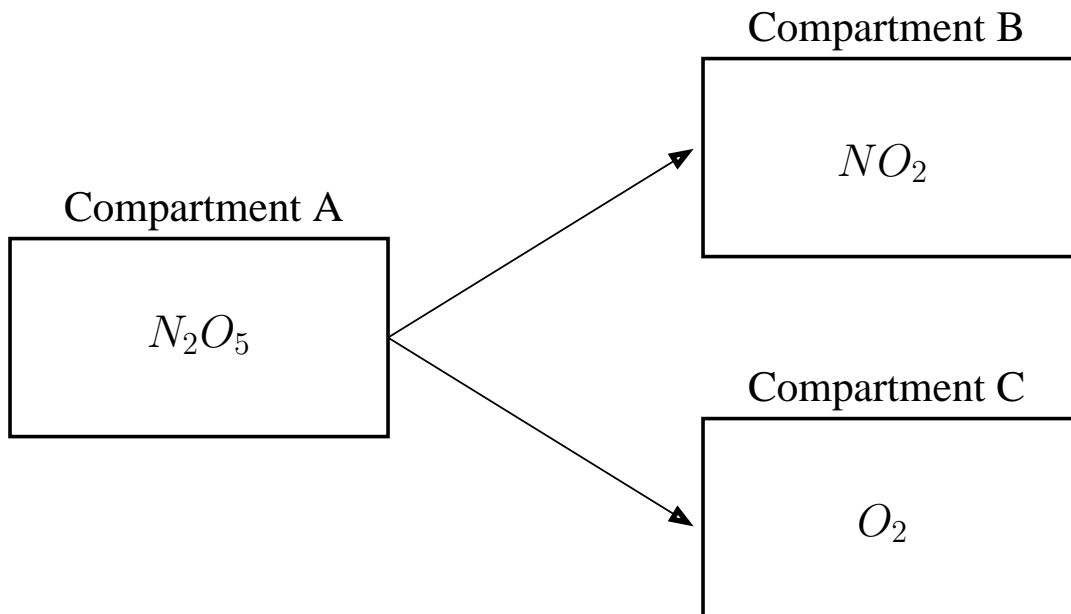


Figure 1.6: Compartments in reaction.

1.5 Chemical Reactions

Compartment models are often used to describe the outcome of chemical reactions. Where conservation of mass is one of the fundamental principles. For example, let us study the following reaction in a solution with volume V_0 :



If we focus on oxygen, we can think of this system as one consisting of three components: A, components on the form N_2O_5 , B, components on the form NO_2 , and C, components on the form O_2 . The connection between the three components (each included in a separate compartment) is shown in Figure 1.6.

The amount of material (in this case the number of oxygen molecules) in compartment A is given by $M_A = V_0[N_2O_5]$ where the brackets indicate the concentration of the material in question, for example in moles/l, the number "5" indicates that one atom of N_2O_5 contain 5 oxygen molecules. The amount of material in the other compartments can be computed similarly.

The velocity of the reaction in (1.6), i.e. the velocity with which one molecule of N_2O_5 is transformed into NO_2 and O_2 , is proportional to the amount of N_2O_5 :

$$\frac{d}{dt}(V_0[N_2O_5]) = -kV_0[N_2O_5] \quad (1.7)$$

where the factor of proportionality is given by k^1 . The number of oxygen molecules can be found by converting the equation above to an equation for the number of oxygen molecules in compartment A (M_A):

$$\frac{d}{dt}M_A = -kM_A. \quad (1.8)$$

In other words, the oxygen that leaves compartment A flows into compartments B and C. The reaction tells us that 4/5 of the oxygen molecules flow into B as NO_2 , while 1/5 of the oxygen molecules flow into C. For the two other compartments, the following equations are obtained:

$$\frac{d}{dt}M_B = \frac{4}{5}kM_A \quad (1.9)$$

$$\frac{d}{dt}M_C = \frac{1}{5}kM_A \quad (1.10)$$

Since we focused on the concentration of oxygen on the form N_2O_5 (compartment A), the solution to the differential equation in (1.8) is given by

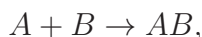
$$M_A(t) = M_0e^{-kt}$$

where M_0 is the amount of oxygen on the form N_2O_5 at the start of the reaction. If we insert this solution into equations (1.9) and (1.10) we get:

$$\begin{aligned} \frac{d}{dt}M_B &= \frac{4}{5}kM_0e^{-kt} \\ \frac{d}{dt}M_C &= \frac{1}{5}kM_0e^{-kt} \end{aligned}$$

Both M_A and M_B can be determined using simple integration on the right hand side of the equation.

While all chemical reactions in principle can be described by compartment models, we will not necessarily get differential equations on the form shown in (1.7). For example if the reaction is given by



then the velocity of the concentration will depend on the product of the two concentrations, as

$$\frac{d}{dt}[A] = -v[A][B]$$

where v is the constant of proportionality, which depend on the incoming chemical reactants. Reactions like the one shown above, where the velocity v of the reaction, depends on the incoming material is called second order reactions. Similarly, reactions where v is proportional to the concentration is called first order reactions.

¹In chemistry it is common to calculate concentrations such as $[N_2O_5]$. The definition of a reaction velocity appears by dividing both sides of the differential equation by the volume V_0

1.6 Limitations of Compartment Models

The basic idea with compartment models is to describe a system as a number of compartments and to derive equations of mass-balance for each of these compartments. In itself writing equations for mass balance is a very healthy technique, which assures that the model has a fundamental background, and that it is possible to judge the validity of the model. However, there is no guarantee that a good compartment model will appear from this exercise.

1.6.1 Is the system closed

The equation for conservation of mass is only correct, if all the material that is added to or removed from the system is described in the model. That is, if the system is closed in some sense. In other words the compartments may not include un-accounted for sources or sinks. Normally, a closed system is obtained by assuming that the total amount of material is constant. Our definition of closed is less strict, since we allow to study systems where material is continuously added or removed. However, in such systems it is crucial to describe how material is added and/or removed.

1.6.2 Is homogeneity a reasonable assumption?

When we use compartment models we assume that all material in the compartment is homogeneous. This assumption can not directly be seen from the equations. Equations only include information about the total amount of material in the compartment, i.e. we cannot include more detailed description of the system. An example could be a compartment model of the blood in the circulatory system. We can describe arteries, veins and the heart, by compartments. These compartments would include blood, for example, it would not be possible to distinguish between plasma, red, and white blood-cells.

It is not always possible to assume homogeneity, for example let us try to use a one-compartment model to describe the concentration of phosphor in a lake. Following this strategy, we assume that the lake has an inflow, an outflow, and a compartment representing the concentration of phosphor. This implies that we assume that all material in the lake is uniformly mixed, and that the amount of phosphor leaving the lake is constant. This is not correct. For example in deep lakes (deeper than 5-10 meter), it is normal to see partition into several layers with different concentration of phosphor. This has to do with the fact that the temperature in the water changes with depth. Often, most of the water flowing into the lake stays at the surface and flow through the lake without exchange with

water in the deeper layers. Hence, we can not calculate the amount of phosphor using a one-compartment model. One way to solve this problem is to let the lake consist of several layers each having a different concentration, and a different amount of phosphor transported between the layers. The disadvantage with splitting the lake into several compartments is that more information is needed to describe the system. For example if the water in the lake is split into two compartments we need to know the concentration of phosphor in the water flowing into and out of the deep layer. This could be very difficult to measure. Even if it is not possible to determine all parameters analytically, it may be advantageous to split the model into smaller subsystem. It is possible using optimization to estimate some unknown parameters, but care must be taken, that enough data exist to obtain realistic parameter values.

1.6.3 Is the balance equation accurate enough?

When the equation of mass balance is derived it is important that all essential transports are known. In real biological systems this is typically not the case. In a realistic system, some of the mass-balances are known but others are not. Before a model can be completed all parameters must be estimated either based on experiments, or calculated based on values found in the literature.

1.6.4 Is the balance of mass relevant?

This last question can seem rather surprising? However, it is not all systems that can be described in terms of mass-balance. This question is easier to understand by discussing a counter example.

In 1950 a compartment model was used to mark rabbits as the biggest threat for survival of the grass-land in Australia. The model was based on observations, that rabbits could eat all grass on an entire field. These observations overestimated the number of rabbits and the amount of damage they could make. The reason for this was that all grass-land was lumped into one compartment independent of their location. As a result the Australian state started to poison rabbits by exposing them to the deadly virus myxomatois, which killed 90% of all rabbits. Today, rabbits are no threat to Australia, and the original study has been criticized for being too crude.

1.6.5 Sensitivity analysis

When a compartment model is derived not all models and initial conditions are known precisely. In addition, the solution to the model will change depending on the model

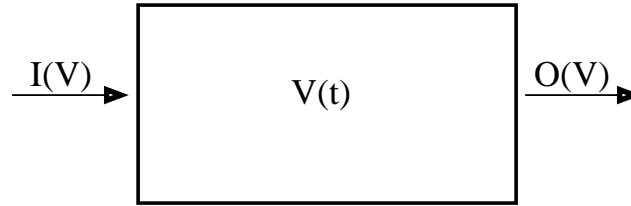


Figure 1.7: Compartment illustrating the volume of fluid in the human body.

parameters and initial conditions. Therefore, it is important to investigate the sensitivity of parameters and initial conditions. This can be done by varying each parameter and record the change in the results.

1.7 Equilibrium Points

In this section we will discuss the qualitative behavior of solutions to compartment model, i.e. we will discuss solutions without computing them quantitatively. We will limit our discussion to solutions that are close to **equilibrium points**.

Assume that we have a compartment with a volume $V(t)$ that varies as a function of time. The compartment has an inflow I and an outflow O (see Figure 1.7). Assume that the both the inflow I and the outflow O depend on V . As an example we consider the balance of fluid in the body. Let $V(t)$ describe the amount of fluid that a given person has at time t . Using a simple model we assume that the intake of fluid I depends on how much fluid is already in the body (i.e. we would not feel thirsty, unless the amount of fluid in the body is low). In other words we assume that $I = I(V)$ and that I is a decreasing function of V (as V increases I decreases). Similarly, we assume that the amount of fluid leaving the body (by respiration, transpiration, and urination) also depend on the amount of fluid in the body $O = O(V)$. The flow out of the body will be an increasing function of V (as the fluid volume increases, so does the outflow). This behavior of I and O is illustrated in Figure 1.8. From the compartment in Figure 1.7 we can derive the following differential equation describing the balance of fluid in the body

$$\frac{dV}{dt} = I - O$$

If a solution $V(t)$ to the differential equation exist, that does not depend on t , i.e. $V(t) = V_0$ is constant, then this solution is called the **equilibrium solution**. In other words, if $V(t)$ is in equilibrium, then the in- and outflow is in equilibrium, and as a result the amount of fluid in the body does not change. If the solution is in equilibrium, then the derivative $dV/dt = 0$ and $V(t) = V_0$. Since both I and O depend on V this must imply that $I(V_0) = O(V_0)$.

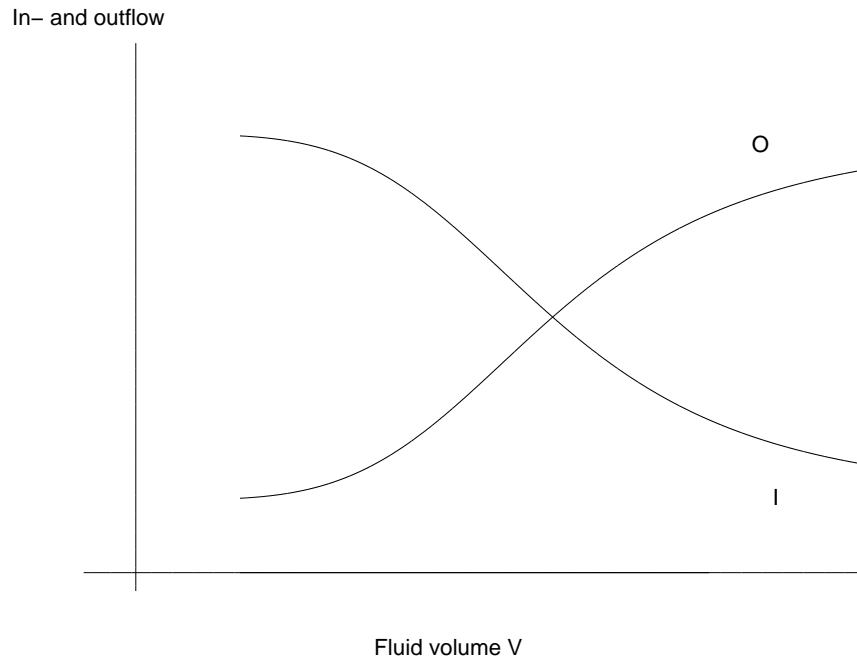


Figure 1.8: Graphs representing inflow I and outflow O .

Since V is monotonically increasing and I is monotonically decreasing only one point can fulfill this condition. This point will be the volume where the two solutions cross each other. In this example only one such point can exist, however, depending on the form of I and O several such points may exist.

Consider the equality point $V(t) = V_0$. If the system is perturbed a small amount ϵ to the right $V(t) = V_0 + \epsilon$ then Figure 1.8 shows that the outflow is increased, i.e. $O(V_0 + \epsilon) > O(V_0)$ and that the inflow is decreased, i.e. $I(V_0 + \epsilon) < I(V_0)$. As a result $I - O$ becomes negative and hence dV/dt becomes negative resulting in a decrease of V . This happens for any small epsilon bigger than zero. As a result V will decrease and get back to V_0 . Similarly, if the system is perturbed to the left, i.e. $V = V_0 - \epsilon$. In this case the outflow is decreased $O(V_0 - \epsilon) < O(V_0)$ and the inflow is increased $I(V_0 - \epsilon) > I(V_0)$. Consequently, $dV/dt > 0$ and the volume will be increased and the equilibrium point V_0 will be reached again. Generally, if an equilibrium point exist and if any perturbation of the system makes the solution go back towards the equilibrium point, then the equilibrium point is said to be **stable**. This is marked on Figure 1.9 with arrows on the x -axis. If the opposite happens, that a perturbation of the solution does not make the solution go back to the equilibrium point, then the point is said to be **unstable**.

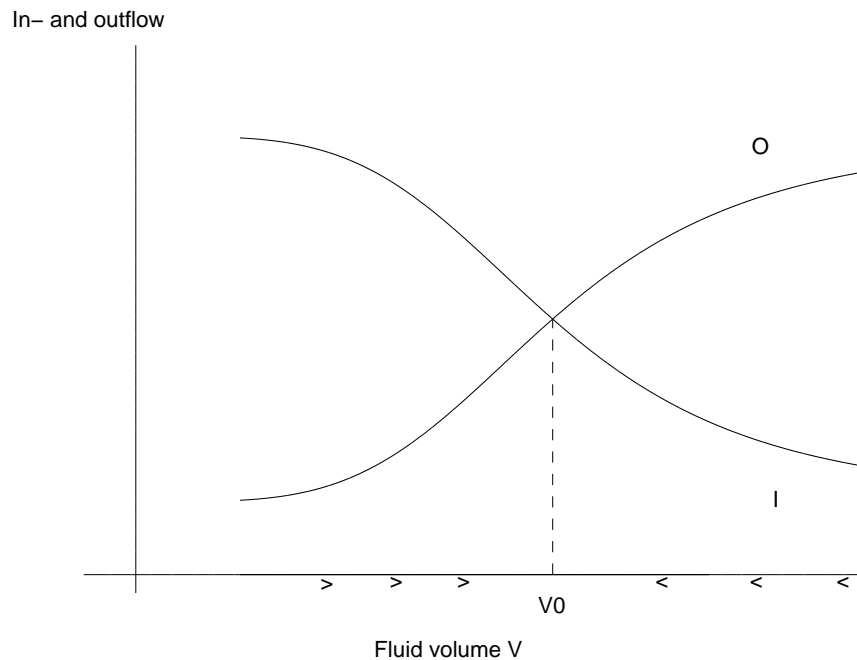


Figure 1.9: Graphs representing inflow I and outflow V . The arrows on the x -axis illustrates stability of the system.

1.8 Exercises

1. Assume that $I = I(V)$ and $O = O(V)$ given in the example above are switched, i.e. that I is increasing function and that O is a decreasing function. Determine possible equilibrium points if any, and determine their stability.
2. Set up a compartment model for a lake with inflow from a creek and from rain and outflow into a sewage system and into another creek. Setup all equations and determine realistic parameters for inflow and outflow.
3. A container has at the start of an experiment 200 ml of fluid. 5 ml of fluid is added to the container with a velocity of 5 ml/min, at the same time, fluid is flowing out from the container at a rate of 8 ml/min.
 - (a) Write up an expression for the volume of the fluid $V(t)$.
 - (b) At some point in time the volume becomes negative, what is wrong?
4. An amount of sodium chloride NaCl is poured into a bucket with clean water. In this solution sodium chloride is transferred to Na^+ and Cl^- with a rate k_1 . At the same time Na^+ reacts with Cl^- to form NaCl with a rate k_2 . Set up a compartment model that describes this situation.